**Relational semantics of intuitionistic logics**

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**Introduction**

Intuitionistic logic, sometimes more generally called constructive logic, refers to systems of symbolic logic that differ from the systems used for classical logic by more closely mirroring the notion of constructive proof. In particular, systems of intuitionistic logic do not include the law of the excluded middle and double negation elimination, which are fundamental inference rules in classical logic.

Formalized intuitionistic logic was originally developed by Arend Heyting to provide a formal basis for L. E. J. Brouwer's programme of intuitionism. From a proof-theoretic perspective, Heyting’s calculus is a restriction of classical logic in which the law of excluded middle and double negation elimination have been removed. Excluded middle and double negation elimination can still be proved for some propositions on a case-by-case basis, however, but do not hold universally as they do with classical logic.

Semantics for intuitionistic modal logic has been considered mainly in Kripke-style, but there is a difficulty in such an approach: the usual interpretation of ♦ (and ∨) validates the distributivity. This is roughly because existential quantification, which is used to interpret ♦ in the meta-level, distributes over disjunction. So, to avoid the distributivity, we need to fix the interpretation of ♦. As a result of such a modification, the analogy between intuitionistic and classical modal logic is lost; this makes intuitionistic modal logics harder to understand as a variant of classical modal logics. For example, consider the correspondences between axioms and properties of frames. One of the simplest example is the axiom p→♦p: classically this axiom corresponds to reflexivity, but in modified Kripke semantics this would not hold.

In classical setting, it is known that a modality ♦ without distributivity cannot be handled in the usual Kripke semantics, and such modalities is said to be non-normal. One of the alternative tools to study such a modality is neighborhood semantics. However, its intuitionistic version has not been extensively studied. Although Sotirov and Wijesekera considers neighborhood semantics for intuitionistic modal logic, it does not seem that they tried to capture the nature of non-distributive ♦ in terms of neighborhood semantics. In Sotirov’s work only a necessity modality is considered, and Wijesekera’s semantics requires some extra axioms for completeness.

**Notation**

PV is a fixed infinite set of propositional variables, and ranged over by p, q.

L(O1, …, On) is the set of formulas built from PV and ⊥ with logical connectives ∧,∨,→ and unary modalities O1, …, On.

¬A is an abbreviation of A → ⊥.

Mostly we will consider L(□,♦) in this paper.

Definition 1.

1. A set Λ⊆L(□,♦) is said to be an L(□,♦)-logic if it contains all intuitionistic tautologies, and closed under uniform substitution, modus ponens, and necessitation (that is, if A∈Λ, then □A∈Λ).

2. Let Λ be an L(□,♦)-logic. A set Λ′⊆L(□,♦) is said to be a Λ-logic if Λ′ is a L(□,♦)-logic and Λ⊆Λ′.

3. Let Λ be an L(□,♦)-logic. A set Γ⊆L(□,♦) is said to be a Λ-theory if it contains Λ and closed under modus ponens.

The following are axioms that will appear in this paper.

(N♦□) ♦⊥ →□⊥ (K) □ (p→q) → □p→□q

(N♦) ¬♦⊥ (K♦) □ (p→q) → ♦p→ ♦q

(PEM) p∨ ¬p

Definition 2. The L(□,♦)-logic IM is the least logic containing K and K♦

Definition 3. Let A be a formula, and Λ be an IM-logic. We define Λ+A as the smallest Λ-logic containing A.

**Existing Relational Approaches**

There have been several approaches to define a Kripke-style semantics for intuitionistic modal logic. Most of them are obtained by introducing an extra structure into the Kripke semantics for intuitionistic logic. Below we will consider a triple <W, ≤, R>. Here ≤ and R are taken from the Kripke semantics for intuitionistic logic and modal logic, respectively. A problem in introducing modalities in this way is that the ordinary truth conditions for modalities

x|Ⱶ □A⇐⇒ ∀y. (x R y=⇒y|Ⱶ A),

x|Ⱶ ♦A⇐⇒ ∃y. (x R y and y|Ⱶ A)

breaks the heredity condition x ≤ y and x|Ⱶ A=⇒y|Ⱶ A, which is expected in the Kripke semantics for intuitionistic logic.

Several solutions to this problem have been proposed in the literature. Roughly speaking, there are two approaches:

1. consider an alternative truth condition, and

2. impose some conditions on ≤ and R,

for each of □ and ♦. For example, Plotkin and Stirling consider 1. For □ and 2. for ♦. This approach results in a logic which admits the distributivity of ♦ over disjunction. Wolter and Zakharyaschev consider 2. For □ and 1. for ♦. In this approach, although the distributivity can be rejected, the duality ♦A↔ ¬□¬A becomes a theorem. Since this is not necessarily natural in intuitionistic setting, we take another approach; we will consider 2. for both □ and ♦. We will define

x |Ⱶ □A⇐⇒ ∀z ≥ x. ∀y. (z R y=⇒y|Ⱶ A),

x|Ⱶ ♦A⇐⇒ ∀z ≥ x. ∃y. (z R y and y|Ⱶ A).

This is the choice often taken in the previous literature to model a ♦ without distributivity.

The above truth condition for ♦ may look strange, because it breaks the analogy between ♦ and ∃. Actually, a ♦ modality without distributivity cannot be interpreted as ∃, because ∃x. (P(x) ∨ Q(x)) implies (∃x. P(x)) ∨ (∃x. Q(x)) in intuitionistic first-order logic. Below, we will summarize known results on the relational semantics for intuitionistic modal logic based on this approach. Basically, the content of the rest of this section is a propositional fragment of Wijesekera’s work.

**Relational Semantic**

Definition 1. An intuitionistic relational frame is a triple <W, ≤, R> of a non-empty set W, a preorder ≤ on W, and a binary relation R on W.

Definition 2.

1. For an intuitionistic relational frame R= <W, ≤, R>, an R-valuation is a map V from PV to P(W).

2. An R-valuation V is said to be admissible if V(p) is upward-closed for all propositional variables p.

Definition 3. An intuitionistic relational model is a pair <R, V> of an intuitionistic relational frame R and an admissible R-valuation V.

Definition 4. Let <R, V> be an intuitionistic relational model. We can define the satisfaction relation, denoted by |Ⱶr, as follows:

R, V, x |Ⱶr p ⇐⇒ x ∈ V(p);

R, V, x |Ⱶr A∧B ⇐⇒ R, V, x |Ⱶr A and R, V, x |Ⱶr B;

R, V, x |Ⱶr A∨B ⇐⇒ R, V, x |Ⱶr A or R, V, x |Ⱶr B;

R, V, x |Ⱶr A→B ⇐⇒ ∀y ≥ x .(R, V, y |Ⱶr A=⇒ R, V, y |Ⱶr B);

R, V, x |Ⱶr □A ⇐⇒ ∀y ≥ x. ∀z. (y R z=⇒ R, V, z |Ⱶr A);

R, V, x |Ⱶr ♦A ⇐⇒ ∀y ≥ x. ∃z. (y R z and R, V, z |Ⱶr A).

Below we sometimes suppress R and V if they are clear from the context.

It is easy to verify that the heredity condition

R, V, x |Ⱶr A and x ≤ y =⇒ R, V, y |Ⱶr A

holds for all formulas A ∈ L(□,♦).

Definition 5. Let A be a formula in L(□,♦).

1. Let <R, V> be an intuitionistic relational model. A is said to be true in <R, V> if R, V, x |Ⱶr A for all x ∈ W.

2. Let K be a class of intuitionistic relational models. A is said to be true in K if it is true in all models of K.

Theorem (Soundness and Completeness). A ∈ L(□,♦) is a theorem of IM + N♦ if and only if A is true in all intuitionistic relational models. Soundness is proved by routine induction, and the completeness can be proved by canonical model construction.